A Combined LP and QP Relaxation for MAP

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MAP Inference for General Pairwise Energies

Energy Minimization

Discrete pairwise energy minimization for graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$:

$$\min_{\mathbf{x}} \sum_{i \in \mathcal{V}} \theta_i(x_i) + \sum_{(i,j) \in \mathcal{E}} \theta_{ij}(x_i, x_j) \qquad x_i \in \{1, \ldots, K\}.$$

For general $\mathcal{G}$ and θ this is an NP-hard problem.

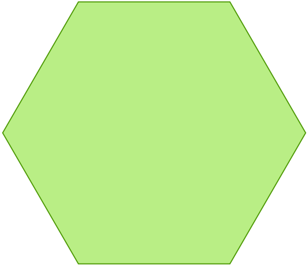
Linear and Quadratic Programming Relaxations

Ravikumar and Lafferty, 2006; Wainwright and Jordan, 2008

LP for Marginal Polytope

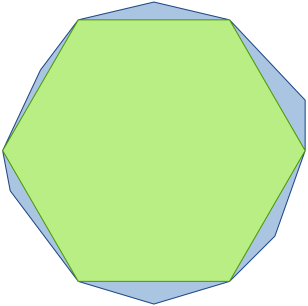
$$\min_{\boldsymbol{\mu} \in \mathcal{M}_{\mathcal{G}}} \quad \sum_{i \in \mathcal{V}} \boldsymbol{\theta}_i^\mathsf{T} \boldsymbol{\mu}_i + \sum_{(i,j) \in \mathcal{E}} \boldsymbol{\theta}_{ij}^\mathsf{T} \boldsymbol{\mu}_{ij}$$

But: $\mathcal{M}_{\mathcal{G}}$ is exponentially large



Linear and Quadratic Programming Relaxations

Ravikumar and Lafferty, 2006; Wainwright and Jordan, 2008



LP for Marginal Polytope

$$\min_{\boldsymbol{\mu} \in \mathcal{M}_{\mathcal{G}}} \quad \sum_{i \in \mathcal{V}} \boldsymbol{\theta}_i^{\mathsf{T}} \boldsymbol{\mu}_i + \sum_{(i,j) \in \mathcal{E}} \boldsymbol{\theta}_{ij}^{\mathsf{T}} \boldsymbol{\mu}_{ij}$$

But: $\mathcal{M}_{\mathcal{G}}$ is exponentially large

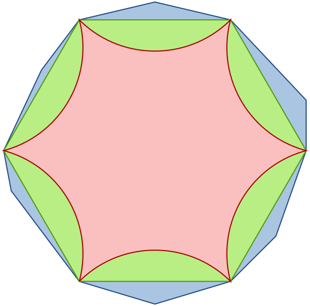
outer approximation

LP for Local Marginal Polytope

$$\min_{\boldsymbol{\mu} \in \mathcal{L}_{\mathcal{G}}} \quad \sum_{i \in \mathcal{V}} \boldsymbol{\theta}_i^{\mathsf{T}} \boldsymbol{\mu}_i + \sum_{(i,j) \in \mathcal{E}} \boldsymbol{\theta}_{ij}^{\mathsf{T}} \boldsymbol{\mu}_{ij}$$

Linear and Quadratic Programming Relaxations

Ravikumar and Lafferty, 2006; Wainwright and Jordan, 2008



LP for Marginal Polytope

$$\min_{\boldsymbol{\mu} \in \mathcal{M}_{\mathcal{G}}} \quad \sum_{i \in \mathcal{V}} \boldsymbol{\theta}_i^{\mathsf{T}} \boldsymbol{\mu}_i + \sum_{(i,j) \in \mathcal{E}} \boldsymbol{\theta}_{ij}^{\mathsf{T}} \boldsymbol{\mu}_{ij}$$

But: $\mathcal{M}_{\mathcal{G}}$ is exponentially large

outer approximation

inner approximation

LP for Local Marginal Polytope

$$\min_{\boldsymbol{\mu} \in \mathcal{L}_{\mathcal{G}}} \quad \sum_{i \in \mathcal{V}} \boldsymbol{\theta}_i^{\mathsf{T}} \boldsymbol{\mu}_i + \sum_{(i,j) \in \mathcal{E}} \boldsymbol{\theta}_{ij}^{\mathsf{T}} \boldsymbol{\mu}_{ij}$$

Quadratic Programming

$$\min_{\boldsymbol{\mu} \in \mathcal{L}_{\mathcal{G}}} \quad \sum_{i \in \mathcal{V}} \boldsymbol{\theta}_i^{\mathsf{T}} \boldsymbol{\mu}_i + \sum_{(i,j) \in \mathcal{E}} \boldsymbol{\theta}_{ij}^{\mathsf{T}} \boldsymbol{\mu}_{ij}$$

$$\text{s.t.} \quad \boldsymbol{\mu}_{ij} = \text{vec}(\boldsymbol{\mu}_i \boldsymbol{\mu}_j^{\mathsf{T}}) \quad \forall (i,j) \in \mathcal{E}$$

Our Approach in a Nutshell

- Combine Linear and Quadratic Programming relaxations.
- Kullback-Leibler penalty term to discourage inconsistencies.
- Solved using an efficient belief-propagation algorithm.

Combined Formulation

Joint Objective

$$\min_{\boldsymbol{\mu} \in \mathcal{L}_{\mathcal{G}}} \underbrace{\sum_{i \in \mathcal{V}} \boldsymbol{\theta}_i^{\mathsf{T}} \boldsymbol{\mu}_i + \sum_{(i,j) \in \mathcal{E}} \boldsymbol{\theta}_{ij}^{\mathsf{T}} \boldsymbol{\mu}_{ij}}_{\text{standard LP objective}} + \underbrace{\beta \sum_{(i,j) \in \mathcal{E}} D_{KL}(\boldsymbol{\mu}_{ij}, \text{vec}(\boldsymbol{\mu}_i \boldsymbol{\mu}_j^{\mathsf{T}}))}_{\text{penalty term for inconsistencies}} .$$

- Constraint set: local marginal polytope $\mathcal{L}_{\mathcal{G}}$.
- Kullback-Leibler divergence as the penalty term:

$$D_{KL}(\boldsymbol{\mu}, \boldsymbol{\nu}) = \sum_k \mu_k \log \left(\frac{\mu_k}{\nu_k} \right) .$$

- Limit cases for β
  - $\beta = 0$: LP relaxation
  - $\beta \to \infty$: QP relaxation.
- Non-convex formulation.

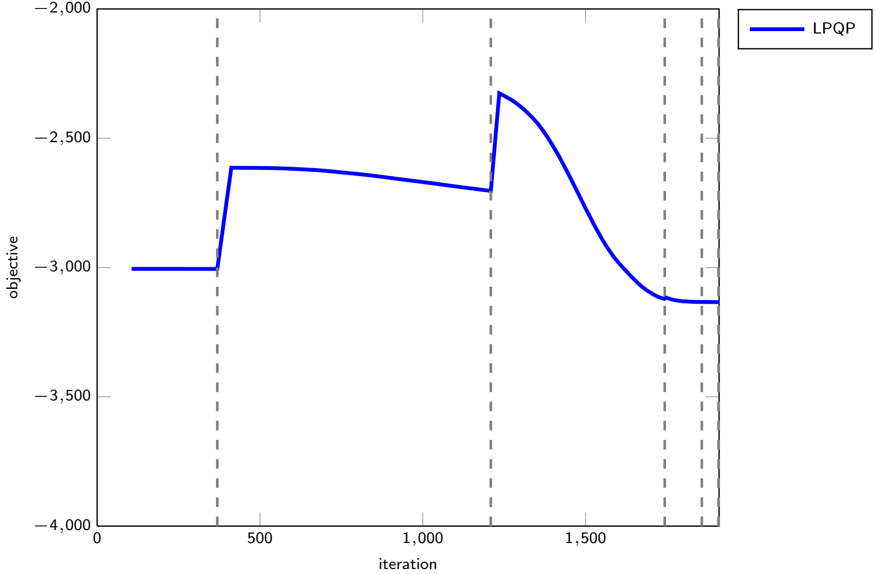
LPQP Algorithm Highlights

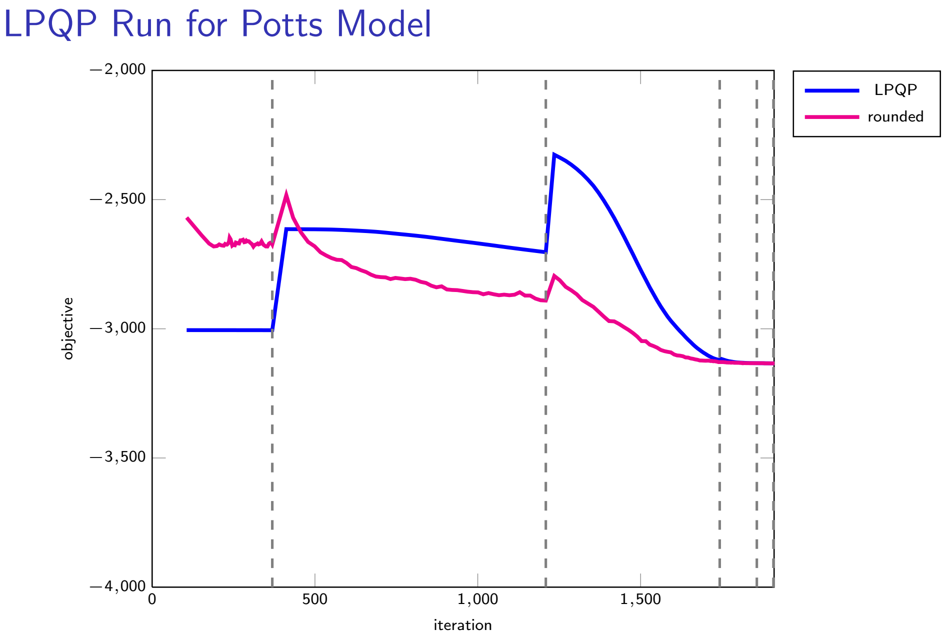
- Decompose objective into a difference of convex functions.
- The main computational task is the following convex problem

$$\min_{\boldsymbol{\mu} \in \mathcal{L}_{\mathcal{G}}} \underbrace{\sum_{i \in \mathcal{V}} \tilde{\boldsymbol{\theta}}_i^{\mathsf{T}} \boldsymbol{\mu}_i + \sum_{(i,j) \in \mathcal{E}} \boldsymbol{\theta}_{ij}^{\mathsf{T}} \boldsymbol{\mu}_{ij}}_{\text{modified LP relaxation}} - \beta \underbrace{\sum_{(i,j) \in \mathcal{E}} H(\boldsymbol{\mu}_{ij})}_{\text{pairwise entropy}} .$$

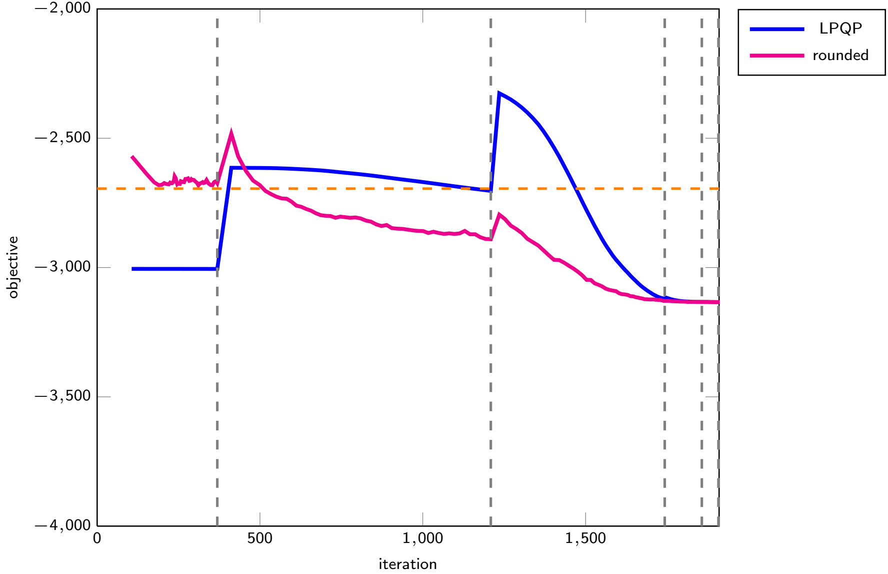
- Solved by norm-product algorithm (Hazan and Shashua, 2010).
- Can warm-start with solution from previous iteration.
- Annealing for β.

LPQP Run for Potts Model




LPQP Run for Potts Model
−2,000
−2,500
−3,000
−3,500
−4,000
0
500
1,000
1,500
objective
iteration
LPQP
rounded

LPQP Run for Potts Model



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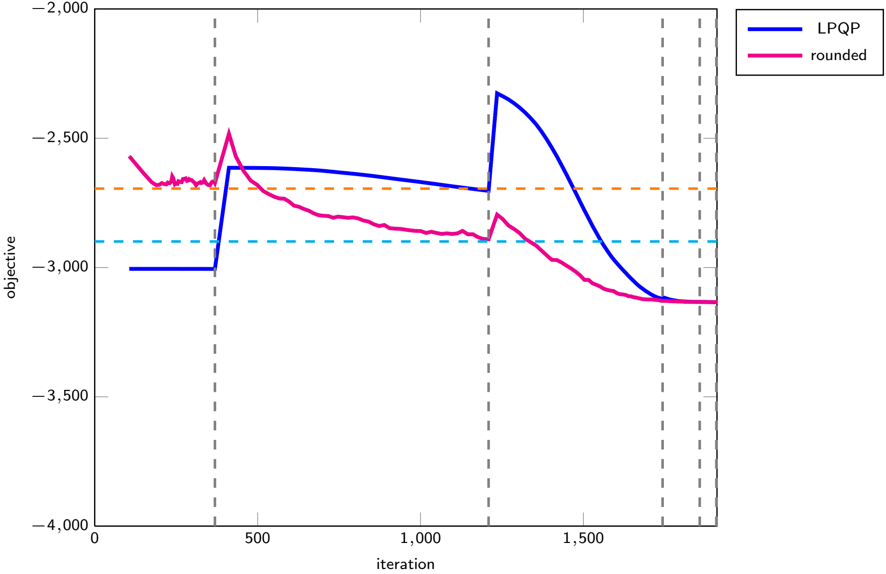

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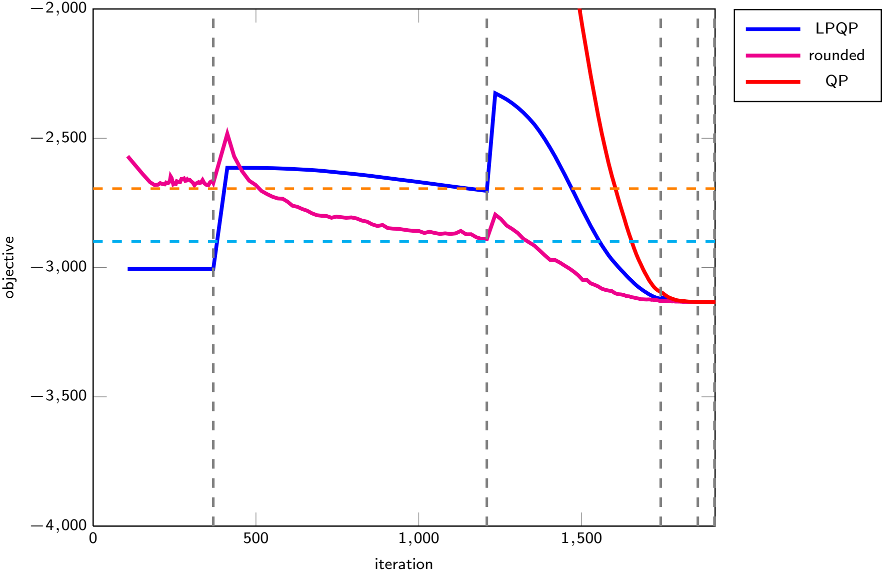
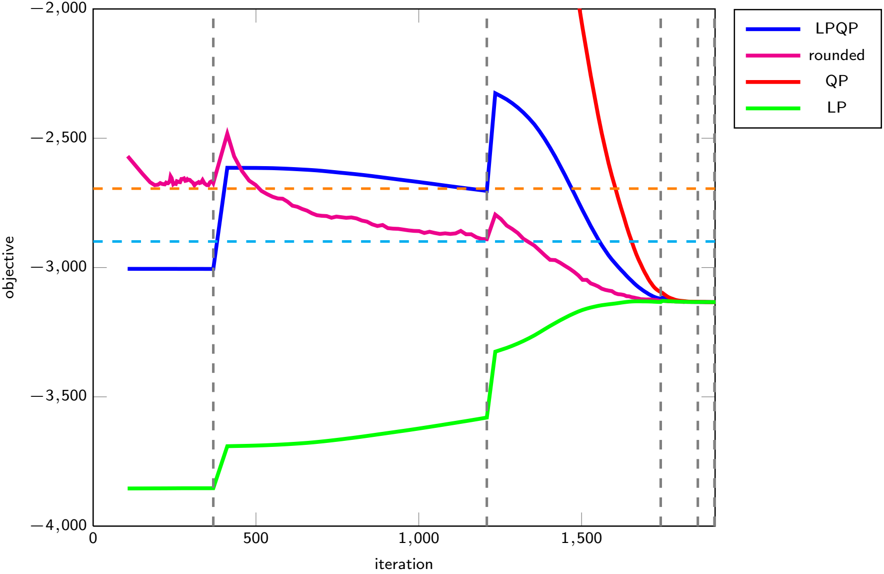


LPQP Run for Potts Model



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Hazan, Tamir and Amnon Shashua (2010). "Norm-Product Belief Propagation: Primal-Dual Message-Passing for Approximate Inference". In: *IEEE Transactions on Information Theory* 56.12, pp. 6294–6316.

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